Exercise 6

Use residues to evaluate the definite integrals in Exercises 1 through 7.

$$\int_0^\pi \frac{d\theta}{(a+\cos\theta)^2} \quad (a>1).$$

$$Ans.\ \frac{a\pi}{\left(\sqrt{a^2-1}\right)^3}.$$

Solution

Notice that the integrand is an even function of θ, so the lower limit of integration can be extended to $-\pi$ as long as the integral is divided by 2.

$$\int_0^\pi \frac{d\theta}{(a+\cos\theta)^2} = \frac{1}{2}\int_{-\pi}^{\pi} \frac{d\theta}{(a+\cos\theta)^2}$$

Now make the substitution,

$$\begin{aligned} \alpha &= \theta+\pi \quad \rightarrow \quad \theta = \alpha - \pi \\ d\alpha &= d\theta, \end{aligned}$$

so that the integral goes from 0 to 2π.

$$\frac{1}{2}\int_{-\pi}^{\pi} \frac{d\theta}{(a+\cos\theta)^2} = \frac{1}{2}\int_{-\pi+\pi}^{\pi+\pi} \frac{d\alpha}{[a+\cos(\alpha-\pi)]^2} = \frac{1}{2}\int_0^{2\pi} \frac{d\alpha}{(a-\cos\alpha)^2}$$

The integral can now be thought of as one over the unit circle in the complex plane.

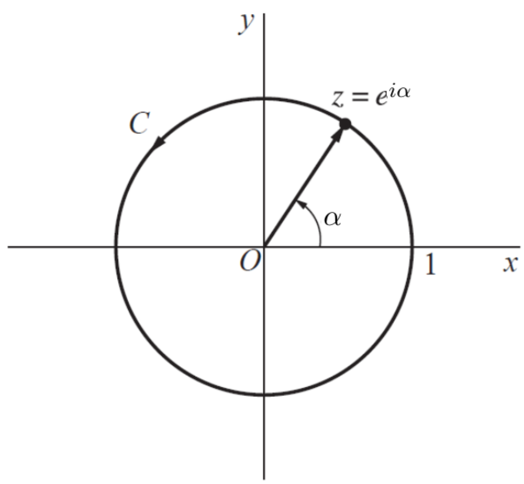

Figure 1: This figure illustrates the unit circle in the complex plane, where $z = x + iy$.

This circle is parameterized in terms of α by $z = e^{i\alpha} = \cos\alpha + i\sin\alpha$. Solve for $\cos\alpha$ and $d\alpha$ in terms of z and dz, respectively.

$$\begin{cases} z = e^{i\alpha} = \cos\alpha + i\sin\alpha \\ z^{-1} = e^{-i\alpha} = \cos\alpha - i\sin\alpha \end{cases} \quad \rightarrow \quad z + z^{-1} = 2\cos\alpha \quad \rightarrow \quad \cos\alpha = \frac{z+z^{-1}}{2}$$

$$z = e^{i\alpha} \quad \rightarrow \quad dz = ie^{i\alpha}\,d\alpha = iz\,d\alpha \quad \rightarrow \quad d\alpha = \frac{dz}{iz}$$

With this change of variables the integral in $d\alpha$ will become a positively oriented closed loop integral over the circle's boundary C.

$$\begin{aligned}\frac{1}{2}\int_0^{2\pi}\frac{d\alpha}{(a-\cos\alpha)^2}&=\oint_C\frac{1}{2}\frac{1}{\left(a-\frac{z+z^{-1}}{2}\right)^2}\frac{dz}{iz}\\&=\oint_C\frac{1}{2}\frac{4z^2}{(z^2-2az+1)^2}\frac{dz}{iz}\\&=\oint_C\frac{-2iz}{(z^2-2az+1)^2}\,dz\end{aligned}$$

According to the Cauchy residue theorem, such an integral in the complex plane is equal to $2\pi i$ times the sum of the residues inside C. Determine the two singular points of the integrand by solving for the roots of the denominator.

$$(z^2-2az+1)^2=0$$
$$z^2-2az+1=0$$
$$z=\frac{2a\pm\sqrt{4a^2-4}}{2}=a\pm\sqrt{a^2-1}\quad\rightarrow\quad\begin{cases}z_1=a+\sqrt{a^2-1}\\z_2=a-\sqrt{a^2-1}\end{cases}$$

Since $a>1$, there is only one singular point inside the unit circle, namely $z=z_2$, so there is only one residue to calculate.

$$\oint_C\frac{-2iz}{(z^2-2az+1)^2}\,dz=2\pi i\operatorname*{Res}_{z=z_2}\frac{-2iz}{(z^2-2az+1)^2}$$

The denominator can be factored as $(z^2-2az+1)^2=(z-z_1)^2(z-z_2)^2$. From this we see that the multiplicity of the factor $z-z_2$ is 2, so the residue is calculated by

$$\operatorname*{Res}_{z=z_2}\frac{-2iz}{(z^2-2az+1)^2}=\frac{\phi^{(2-1)}(z_2)}{(2-1)!}=\phi'(z_2),$$

where $\phi(z)$ is the same function as the integrand without the factor $(z-z_2)^2$.

$$\phi(z)=\frac{-2iz}{(z-z_1)^2}$$

So then

$$\operatorname*{Res}_{z=z_2}\frac{-2iz}{(z^2-2az+1)^2}=-\frac{ia}{2\left(\sqrt{a^2-1}\right)^3}$$

and

$$\oint_C\frac{-2iz}{(z^2-2az+1)^2}\,dz=2\pi i\left[-\frac{ia}{2\left(\sqrt{a^2-1}\right)^3}\right]=\frac{a\pi}{\left(\sqrt{a^2-1}\right)^3}.$$

Therefore,

$$\int_0^{\pi}\frac{d\theta}{(a+\cos\theta)^2}=\frac{a\pi}{\left(\sqrt{a^2-1}\right)^3}\quad(a>1).$$